

# NEW PROOFS OF BORWEIN-TYPE ALGORITHMS FOR $\pi$

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**ABSTRACT.** We use a method of translation to recover Borweins' quadratic and quartic iterations. Then, by using the WZ-method, we obtain some initial values which lead to the limit  $1/\pi$ . We will not use the modular theory nor either the Gauss' formula that we used in [7]. Our proofs are short and self-contained.

## 1. INTRODUCTION

The complete elliptic integral  $K(x)$  is defined as follows:

$$K(x) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - x^2 \sin^2 t}},$$

and its expansion in power series is given by

$$\frac{2}{\pi} K(x) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2} \middle| x^2\right) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} x^{2k},$$

where  $(a)_k = a(a+1) \cdots (a+k-1)$  is the rising factorial, also known as Pochhammer symbol. Following the Indian mathematical genius Srinivasa Ramanujan, an identity of the form

$$\frac{K(\sqrt{1-v^2})}{K(v)} = n \frac{K(\sqrt{1-u^2})}{K(u)},$$

where  $n$  is any positive rational number, induces an algebraic transformation of degree  $n$ , which is of the following form [9] or [1, Ch. 29]:

$$K(u) = m(u, v) K(v), \quad P(u, v) = 0,$$

where  $m(u, v)$  (the multiplier) and  $P(u, v)$  (the modular equation) are algebraic functions in  $u$  and  $v$ . Many modular equations stated by Ramanujan in the variables  $\alpha = u^2$  and  $\beta = v^2$  are proved by Berndt in [2, Ch.19]. In addition, Ramanujan proved that if  $u_0$  is a solution of the equation  $P(u, \sqrt{1-u^2}) = 0$ , then there exist two algebraic numbers  $C_1$  and  $C_2$  such that

$$(1) \quad \left( C_1 \frac{K^2}{\pi^2} + C_2 \frac{KK'}{\pi^2} \right) (u_0) = \frac{1}{\pi},$$

Observe that as  $K$  is a common factor, we can write (1) in the following way

$$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} u_0^{2k} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} u_0^{2k} (A + Bk) = \frac{1}{\pi},$$

where  $A$  and  $B$  are algebraic numbers related to  $u_0$ ,  $C_1$  and  $C_2$ . Instead, in his proof Ramanujan relates (1) to a unique  ${}_3F_2$  hypergeometric sum of the following form:

$$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{(1)_k^3} [4u_0^2(1-u_0^2)]^k (a+bk) = \frac{1}{\pi},$$

where  $a$ ,  $b$  are algebraic numbers. One example given by Ramanujan is

$$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{(1)_k^3} \frac{5+42k}{64^k} = \frac{16}{\pi}.$$

In the middle eighties, the brothers Jonathan and Peter Borwein observed that the modular equations given by Ramanujan could be connected to the number  $\pi$  via the *elliptic alpha function*, to construct extraordinary rapid algorithms to calculate  $\pi$  [3, p.170]. The point of view of the Borweins' is explained in [4], and their proofs require a good knowledge of the elliptic modular functions. In this paper we show a different point of view, and although the idea of our method looks very general, we only focus in a quadratic transformation, which will allow us to recover the quadratic and quartic algorithms for a few sets of initial values (got with the WZ-method). The main point is that we avoid completely the use of the modular stuff, and also the Gauss's formula used in [7].

## 2. TRANSFORMATIONS AND ALGORITHMS

Applying Zudilin's translation method [10] to a quadratic transformation we will derive Borweins' quadratic iteration.

**2.1. A quadratic transformation.** The following identity:

$$(2) \quad \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} x^{2k} = (1+t) \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} t^{2k}.$$

where

$$(3) \quad t = \frac{1 - \sqrt{1-x^2}}{1 + \sqrt{1-x^2}},$$

and  $0 < x < 1$  is a known quadratic transformation [3, Thm 1.2 (b)]. Observe that the transformation is quadratic because  $t \sim 4^{-1} \cdot x^2$ . To prove (2) just expand in powers of  $x$ , observe that the coefficients up to  $x^2$  are equal and check that both sides of (2) satisfy the differential equation:

$$((x^2 - 1)\vartheta_x^2 + 2x^2\vartheta_x + x^2) f = 0, \quad \vartheta_x = x \frac{d}{dx},$$

taking into account that the recurrence satisfied by the coefficient  $D_k = (1/2)_k^2 / (1)_k^2$  of  $x^{2k}$  and  $t^{2k}$  is:

$$D_0 = 1, \quad D_{k+1} = \frac{(1+2k)^2}{4(1+k)^2} D_k,$$

the following relations which come from (3):

$$\sqrt{1-x^2} = \frac{1-t}{1+t}, \quad x^2 = \frac{4t}{(1+t)^2}, \quad x \frac{dt}{dx} = \frac{x^2(1+t)^3}{2-2t} = \frac{2t(1+t)}{1-t},$$

and

$$\vartheta_x = \frac{x}{t} \frac{dt}{dx} \vartheta_t = \frac{2+2t}{1-t} \vartheta_t, \quad \vartheta_x^2 = \frac{4(1+t)^2}{(1-t)^2} \vartheta_t^2 + \frac{8t(1+t)}{(1-t)^3} \vartheta_t.$$

**2.2. A quadratic algorithm.** We consider the following operator:

$$a + \frac{b}{2} \vartheta_x = a + b \frac{1+t}{1-t} \vartheta_t,$$

and apply it to both sides of (2):

$$\left( a + \frac{b}{2} \vartheta_x \right) \left\{ \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} x^{2k} \right\} = \left( a + b \frac{1+t}{1-t} \vartheta_t \right) \left\{ (1+t) \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} t^{2k} \right\}.$$

We have

$$(4) \quad \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} (a + bk) x^{2k} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} \left( a(1+t) + b \frac{t(1+t)}{1-t} + 2b \frac{(1+t)^2}{1-t} k \right) t^{2k}$$

Multiplying (2) and (4), we obtain

$$(5) \quad \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} x^{2k} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} (a + bk) x^{2k} \\ = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} t^{2k} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} \left( a(1+t)^2 + b \frac{t(1+t)^2}{1-t} + 2b \frac{(1+t)^3}{1-t} k \right) t^{2k}$$

Let

$$A_n = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} d_n^{2k} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} (a_n + b_n k) d_n^{2k}.$$

We define the sequences  $d_n$ ,  $a_n$ ,  $b_n$ ,  $A_n$  of initial values  $d_0$ ,  $a_0$ ,  $b_0$ ,  $A_0$ , implicitly by means of  $A_n = A_{n+1}$ , where  $A_{n+1}$  comes from  $A_n$  in the same way than the right hand side of (5) comes from its left side. Hence we have the following recurrences:

$$(6) \quad d_{n+1} = \frac{1 - \sqrt{1 - d_n^2}}{1 + \sqrt{1 - d_n^2}}, \quad b_{n+1} = 2b_n \frac{(1 + d_{n+1})^3}{1 - d_{n+1}},$$

$$(7) \quad a_{n+1} = a_n(1 + d_{n+1})^2 + \frac{b_{n+1}d_{n+1}}{2(1 + d_{n+1})}.$$

As  $A_n$  is a constant sequence, it implies that  $\lim A_n = A_0$ , and as  $d_n^{2k} \rightarrow 0$  and  $b_n d_n^{2k} \rightarrow 0$  when  $n \rightarrow \infty$  and  $k \neq 0$ , we deduce that  $\lim A_n = \lim a_n$ . Hence  $\lim a_n = A_0$ .

**Remark.** Instead of dealing with algebraic transformations of products of two  ${}_2F_1$  hypergeometric sums, we can deal with transformations of single  ${}_3F_2$  hypergeometric series. In fact, there is an equivalence. For example, from the known transformation

$$\left( \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^2}{(1)_k^2} x^{2k} \right)^2 = \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{(1)_k^3} [4x^2(1-x^2)]^k,$$

we can, via translation, rewrite our products of two  ${}_2F_1$  series as unique  ${}_3F_2$  series. When the sum is equal to  $1/\pi$  these single sums constitute one of the families of Ramanujan-type series for  $1/\pi$ .

### 3. OTHER ALGORITHMS

Here we give some related algorithms:

**3.1. A simplified version.** From (6), we have

$$\frac{b_{n+1}}{1-d_{n+1}^2} = 2b_n \frac{(1+d_{n+1})^2}{(1-d_{n+1})^2} = 2 \frac{b_n}{1-d_n^2}.$$

Then, writing  $c_n = b_n/(1-d_n^2)$ , we have  $c_{n+1} = 2c_n$ , which imply  $c_n = c_0 \cdot 2^n$ , and we get a simplified version of the algorithm of subsection 2.2. It is

$$(8) \quad d_{n+1} = \frac{1 - \sqrt{1-d_n^2}}{1 + \sqrt{1-d_n^2}},$$

$$(9) \quad a_{n+1} = a_n(1+d_{n+1})^2 + c_0 2^n d_{n+1}(1-d_{n+1}),$$

with  $\lim a_n = A_0$ .

**3.2. An even more simplified version: Borweins' quadratic algorithm.** As  $c_0 2^n d_{n+1} \sim a_{n+1} - a_n$ , we deduce that  $2^n d_{n+1}$  tends to 0 as  $n \rightarrow \infty$ . Then, writing  $a_n = r_n - c_0 2^{n-1} d_n^2$ , we arrive to the Borweins' quadratic iteration

$$(10) \quad d_{n+1} = \frac{1 - \sqrt{1-d_n^2}}{1 + \sqrt{1-d_n^2}},$$

$$(11) \quad r_{n+1} = r_n(1+d_{n+1})^2 - c_0 2^n d_{n+1},$$

with  $\lim r_n = \lim a_n = A_0$ .

**3.3. Borweins' quartic algorithm.** Making  $s_n = \sqrt{d_{2n}}$ ,  $t_n = r_{2n}$ , we can arrive to the following Borweins' quartic algorithm (see [7] for the steps):

$$\begin{aligned} s_0 &= \sqrt{d_0}, \quad t_0 = r_0, \\ s_{n+1} &= \frac{1 - \sqrt[4]{1-s_n^4}}{1 + \sqrt[4]{1-s_n^4}}, \\ t_{n+1} &= (1+s_{n+1})^4 t_n - c_0 2^{2n+1} s_{n+1}(1+s_{n+1}+s_{n+1}^2), \end{aligned}$$

with  $\lim t_n = A_0$ . Observe that it is a quartic algorithm because  $s_{n+1} \sim 8^{-1} \cdot s_n^4$ .

**3.4. Initial values.** The identities of this subsection provide initial values in order that the iterations tend to  $A_0 = 1/\pi$ . Below each identity we show the corresponding initial values in the algorithms. We will prove these evaluations in next subsection.

*Identity 1.*

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^2}{(1)_k^2} (3 - 2\sqrt{2})^{2k} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^2}{(1)_k^2} (3 - 2\sqrt{2})^{2k} \left[ 16(3\sqrt{2} - 4)k + 4(5\sqrt{2} - 7) \right] = \frac{1}{\pi},$$

and

$$d_0 = s_0^2 = 3 - 2\sqrt{2}, \quad b_0 = 48\sqrt{2} - 64, \quad a_0 = 20\sqrt{2} - 28, \quad c_0 = 4, \quad r_0 = t_0 = 6 - 4\sqrt{2}.$$

*Identity 2.*

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^2}{(1)_k^2} \frac{1}{2^k} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^2}{(1)_k^2} \frac{k}{2^k} = \frac{1}{\pi},$$

and

$$d_0 = s_0^2 = \frac{1}{\sqrt{2}}, \quad b_0 = 1, \quad a_0 = 0, \quad c_0 = 2, \quad r_0 = t_0 = \frac{1}{2}.$$

*Identity 3.*

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^2}{(1)_k^2} (\sqrt{2} - 1)^{2k} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^2}{(1)_k^2} (\sqrt{2} - 1)^{2k} \left[ (8 - 4\sqrt{2})k + (3 - 2\sqrt{2}) \right] = \frac{1}{\pi},$$

and

$$d_0 = s_0^2 = \sqrt{2} - 1, \quad b_0 = 8 - 4\sqrt{2}, \quad a_0 = 3 - 2\sqrt{2}, \quad c_0 = 2\sqrt{2}, \quad r_0 = t_0 = \sqrt{2} - 1.$$

*Identity 4.*

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^2}{(1)_k^2} \left( \frac{2 - \sqrt{3}}{4} \right)^k \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^2}{(1)_k^2} \left( \frac{2 - \sqrt{3}}{4} \right)^k \left[ \left( \frac{3}{2} + \sqrt{3} \right) k + \frac{1}{4} \right] = \frac{1}{\pi},$$

and

$$d_0 = s_0^2 = \frac{\sqrt{6} - \sqrt{2}}{4}, \quad b_0 = \frac{3}{2} + \sqrt{3}, \quad a_0 = \frac{1}{4}, \quad c_0 = 2\sqrt{3}, \quad r_0 = t_0 = \frac{\sqrt{3} - 1}{2}.$$

*Identity 5.*

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^2}{(1)_k^2} (2\sqrt{2} - 2)^k \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^2}{(1)_k^2} (2\sqrt{2} - 2)^k \left[ (3\sqrt{2} - 4)k + \sqrt{2} - \frac{3}{2} \right] = \frac{1}{\pi},$$

and

$$d_0 = s_0^2 = \sqrt{2\sqrt{2} - 2}, \quad b_0 = 3\sqrt{2} - 4, \quad a_0 = \sqrt{2} - \frac{3}{2}, \quad c_0 = \sqrt{2}, \quad r_0 = t_0 = \frac{1}{2}.$$

**3.5. Elementary proofs of identities 1 to 5.** In this section we prove Identities 1 to 5. We avoid the modular theory and use only the WZ-method together with Carlson's Theorem [5]. In the right side of the formulas we use the general definition of the Pochhammer symbol, namely  $(v)_k = \Gamma(v + k)/\Gamma(v)$ .

*WZ-proof of Identity 1.* The following identity:

$$(12) \quad \sum_{n=0}^{\infty} \frac{(\frac{1}{2} - 4k)_n (\frac{1}{2} - 2k)_n}{(1 + 2k)_n (1)_n} (3 - 2\sqrt{2})^{2n} = C_1 (3\sqrt{2} - 4)^{2k} \frac{2^{9k}}{3^{3k}} \frac{(1)_k (\frac{1}{2})_k}{(\frac{7}{12})_k (\frac{11}{12})_k},$$

valid for any complex number  $k$  is WZ-provable [8], due to the fact that it has a free parameter, and we can determine the constant  $C_1$  by taking  $k = 1/4$ . To discover other formulas like this, we need that the function  $H(n, k)$  inside the sum is such that the output of the Maple code

```
with(SumTools[Hypergeometric]);
degree(Zeilberger(H(n,k),k,n,K)[1],K);
```

is 1. In this case we can easily guess the function at the right hand side of the equal, which is a function depending only in  $k$ . Many terminating versions of formulas of this kind have been discovered and proved automatically by a program written by D. Zeilberger [6]. It is good for our purposes that we found the following complement of (12):

$$(13) \quad \sum_{n=0}^{\infty} \frac{(\frac{1}{2} - 4k)_n (\frac{1}{2} - 2k)_n}{(1 + 2k)_n (1)_n} (3 - 2\sqrt{2})^{2n} \left[ 16(3\sqrt{2} - 4)n + 16(13 - 9\sqrt{2})k + 4(5\sqrt{2} - 7) \right] \\ = C_2 (3\sqrt{2} - 4)^{2k} \frac{2^{9k}}{3^{3k}} \frac{(1)_k (\frac{1}{2})_k}{(\frac{1}{12})_k (\frac{5}{12})_k},$$

which is WZ-provable as well. To discover this complement we have used the following Maple code:

```
with(SumTools[Hypergeometric]);
coK2:=coeff(Zeilberger(H(n,k)*(n+b*k+c),k,n,K)[1],K,2);
coes:=coeffs(coK2,k); solve({coes},{b,c});
```

where  $H(n, k)$  is the function inside the sum in (12) written in the notation of Maple. Again, we can determine  $C_2$  taking  $k = 1/4$ , and we have that  $C_1 C_2 = 1/\pi$ . Multiplying the formulas (12) and (13), taking  $k = 0$ , and finally replacing  $n$  with  $k$  to agree with the notation of the other subsections, we arrive at Identity 1.

Below we give couples of identities which prove Identities 2 to 5.

*WZ-proof of Identity 2.*

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2} - 2k)_n (\frac{1}{2} + 2k)_n}{(1 + 4k)_n (1)_n} \left(\frac{1}{2}\right)^n = C_1 \left(\frac{16}{27}\right)^k \frac{(1)_k (\frac{1}{2})_k}{(\frac{7}{12})_k (\frac{11}{12})_k}, \\ \sum_{n=0}^{\infty} \frac{(\frac{1}{2} - 2k)_n (\frac{1}{2} + 2k)_n}{(1 + 4k)_n (1)_n} \left(\frac{1}{2}\right)^n (n + 4k) = C_2 \left(\frac{16}{27}\right)^k \frac{(1)_k (\frac{1}{2})_k}{(\frac{1}{12})_k (\frac{5}{12})_k}.$$

WZ-proof of Identity 3.

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2} - 2k)_n (\frac{1}{2})_n}{(1 + 2k)_n (1)_n} (\sqrt{2} - 1)^{2n} = C_1 \frac{(1)_k (\frac{1}{2})_k}{(\frac{5}{8})_k (\frac{7}{8})_k},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2} - 2k)_n (\frac{1}{2})_n}{(1 + 2k)_n (1)_n} (\sqrt{2} - 1)^{2n} \left[ (8 - 4\sqrt{2})n + (8\sqrt{2} - 8)k + (3 - 2\sqrt{2}) \right] = C_2 \frac{(1)_k (\frac{1}{2})_k}{(\frac{1}{8})_k (\frac{3}{8})_k}.$$

WZ-proof of Identity 4.

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2} - k)_n (\frac{1}{2} + 3k)_n}{(1 + k)_n (1)_n} \left( \frac{2 - \sqrt{3}}{4} \right)^n = C_1 \left( \frac{4}{3\sqrt{3}} \right)^k \frac{(1)_k}{(\frac{5}{6})_k},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2} - k)_n (\frac{1}{2} + 3k)_n}{(1 + k)_n (1)_n} \left( \frac{2 - \sqrt{3}}{4} \right)^n \left[ \left( \frac{3}{2} + \sqrt{3} \right) n + \frac{3}{2}k + \frac{1}{4} \right] = C_2 \left( \frac{4}{3\sqrt{3}} \right)^k \frac{(1)_k}{(\frac{1}{6})_k}.$$

WZ-proof of Identity 5.

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2} + 4k)_n (\frac{1}{2} + 2k)_n}{(1 + k)_n (1)_n} (2\sqrt{2} - 2)^n = C_1 (3 + 2\sqrt{2})^{2k} \frac{(1)_k (\frac{1}{2})_k}{(\frac{5}{8})_k (\frac{7}{8})_k},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2} + 4k)_n (\frac{1}{2} + 2k)_n}{(1 + k)_n (1)_n} (2\sqrt{2} - 2)^n \left[ (3\sqrt{2} - 4)n + (8\sqrt{2} - 12)k + \left( \sqrt{2} - \frac{3}{2} \right) \right]$$

$$= C_2 (3 + 2\sqrt{2})^{2k} \frac{(1)_k (\frac{1}{2})_k}{(\frac{1}{8})_k (\frac{3}{8})_k}.$$

Although with the WZ-method we have only obtained a few sets of initial values, it supposes no disadvantage with respect to the modular theory because the interesting fact is to have one set of simple initial values.

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